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Five Hundred and Thirty-fourth Meeting.

May 10, 1864. — MONTHLY MEETING.

The PRESIDENT in the chair.

The President called the attention of the Academy to the recent decease of Dr. John Ware of the Resident Fellows, and of General Joseph G. Totten of the Associate Fellows.

Mr. Oliver presented a partial investigation on the best approximate representation of all the mutual ratios of k quantities by those of simple integers.

When $k = 2$, it is well known that the method of continued fractions gives every pair of integers whose ratio is more accurate than that of any simpler pair: [as well as every *proper* approximate ratio, i. e. every approximation $\frac{m_1^{(i)}}{m_2^{(i)}}$ to $\frac{m_1}{m_2}$ in which a certain further criterion that involves either the previous or the next proper approximation, as that $m_1^{(i)}\, m_2^{(i+1)} - m_2^{(i)}\, m_1^{(i+1)} = \pm 1$, is satisfied.]

But when $k > 2$, the $\frac{1}{2}\, k\, (k-1)$ separate ratios of the given quantities $(m_1, \ldots m_k)$ can seldom be very accurately represented by those of the same small integers $m'_1, \ldots m'_k$; and a different kind of criterion is needed to show how far we must rectify either ratio at the expense of others. This *criterion of excellence* depends, of course, on the smallness of some such function V of the $\frac{1}{2}\, k\, (k-1)$ determinants $m_r\, m'_s - m_s\, m'_r$* and the $2\, k$ general magnitudes $(m_1, \ldots m'_k)$, as shall be independent of the interchange of (m_r, m'_r) with (m_s, m'_s), and shall vanish when all the determinants do, and not otherwise when $(m_1, \ldots m'_k)$ are real and positive. But it remains to fix the precise *form* of V by other considerations; for instance, by the necessities of some particular problem.

Here the simplest case of the problem "to determine the best approximate weights of a set of linear equations," seems likely to give a very natural criterion, whether others equally natural exist or not. Let $(m_1, \ldots m_k)$ be the weights of equations $(x = x_1, \ldots x = x_k)$, from which to determine x.

Let $\quad S = m_1 + \ldots m_k, \qquad S' = m'_1 + \ldots m'_k.$

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* Usually of the *squared* determinants, so as to be independent of the determinants' *signs*.

If we knew the tensors, but not the signs, of the errors $(e_1, .. e_k)$ of $(x_1, .. x_k)$, the consideration that all the 2^k sign-combinations are equally probable, would give for that average or probable increase in the squared error of x due to the inaccuracy of $(m'_1, : .. m'_k)$, the value

$$\left(\frac{m'_1}{S'}-\frac{m_1}{S}\right)^2 e_1^2 + .. \left(\frac{m'_k}{S'}-\frac{m_k}{S}\right)^2 e_k^2;$$

but since of $(e_1^2, .. e_k^2)$ we know only that their probable values are inversely as $(m_1 : .. m_k)$, on the smallness of

$$\frac{1}{W}=\frac{1}{m_1}\left(\frac{m'_1}{S'}-\frac{m_1}{S}\right)^2 + .. \frac{1}{m_k}\left(\frac{m'_k}{S'}-\frac{m_k}{S}\right)^2$$

must depend the required criterion. Since $\frac{1}{W}$ is here proportioned to the square of the probable effect of the inaccuracy of $(m'_1 : .. m'_k)$ upon x, W may be called the *weight* of the approximation $(m'_1 : .. m'_k)$ to $(m_1 : .. m_k)$.

It remains to examine the properties and results of the criterion $F\,(\frac{1}{W}, m_1, .. m'_k) = \text{min.}$, where F depends merely on the general magnitudes of $\frac{1}{W}$, m_1, &c., ..; and to compare them with those of $F\left(\frac{1}{P}, m_1, .. m'_k\right) = \text{min.}$, &c.

It may be said that Nature, while obeying with absolute precision the resultant of her efficient laws, such as Inertia, Attraction, &c., falls into forms which commonly both *utility* and *taste*, often independently of each other, would prescribe; that some of her æsthetic laws can be learned from the simple integers (or other simple members of a series whose form is prescribed *a priori*), whose ratios nearly represent those of certain natural constants; and that the probability that such æsthetic laws are *intentionally* observed may be roughly computed from the constants. Hence the need of some criterion to guide beforehand our selection of approximate ratios, unbiassed by the interests of the special æsthetic law we are testing.

The phyllotaxis, and still more the planetotaxis noticed by Peirce, illustrate one such æsthetic law. The mutual configurations of two planets, and even their separate motions, whether we regard the disturbed orbits as variable ellipses or as higher curves, are patterns containing the element time; and their character must depend mainly upon those integers whose ratios best represent those of the revolution-periods. This character is probably most peculiar and beautiful, because most distinct, when the period-ratio is

$$\cfrac{1}{n+\cfrac{1}{1+\,..\,\cfrac{1}{1+\cfrac{1}{p}}}} \qquad [n \text{ an integer, } p \text{ large,}]$$

as in the actual solar system; for were the period-ratio

$$\cfrac{1}{n+\,..\,\cfrac{1}{a+\cfrac{1}{b+\,..\,\cfrac{1}{c+\,..}}}} \qquad [n, .. \, a, b, .. \, c, .. \text{ integers, } b > 1,]$$

the character due to the ratio $\left(\cfrac{1}{n+\,..\,\cfrac{1}{c}}\right)$ would be somewhat confused and overlaid by that of the lower ratio $\left(\cfrac{1}{n+\,..\,\cfrac{1}{a}}\right)$. Since p is large, the pattern nearly repeats itself after a few revolutions.

Professor Peirce thought that different discussions would require different criteria of excellence in the representation of the constant-ratios involved; and that it would be of interest to determine that form of V which is smallest for those approximate period-ratios which best represent the aggregate of all "terms of long period" in the theory of two or more planets.

Five hundred and thirty-fifth Meeting.

May 24, 1864. — Annual Meeting.

The President in the chair.

The Corresponding Secretary read letters relative to the exchanges of the Academy; also a letter from Richard H. Dana, Jr., in acknowledgment of his election into the Academy.

The Report of the Council upon the changes which have occurred in the Academy during the past year was presented, and read in part by the Corresponding Secretary, as follows: —